

A hyperbolic parametrization of lepton mixing for small U_{e3}

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Abstract

We use hyperbolic functions to parametrize lepton mixing matrix using only one input parameter ϕ . This matrix has three mixing angles as outputs therefore giving two predictions. In particular it predicts $U_{e3} = 0$ besides predicting correct solar and atmospheric mixing. We confine us to real ϕ . For complex ϕ mixing matrix is no longer unitary. Next we accentuate this unitary mixing matrix with an additional small parameter δ while keeping unitarity of the matrix exact. When this second parameter is included, with this framework, one can handle small but non-zero U_{e3} . In the second case from two input parameters one obtains three mixing angles thus one prediction.

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Heroic advancements in the realm of neutrino physics experiments [1, 2, 3, 4, 5, 6, 7] have given us substantial information on lepton mixing angles. Many different groups have extracted lepton mixing angles from raw data [8, 9, 10, 11, 12, 13, 14]. Using these informations magnitudes of nine elements of this 3×3 unitary mixing matrix U_{lep} approximately looks like [15, 16, 17, 18, 19],

$$|U_{lep}| = \begin{pmatrix} 0.72 - 0.88 & 0.46 - 0.68 & < 0.22 \\ 0.25 - 0.65 & 0.27 - 0.73 & 0.55 - 0.84 \\ 0.10 - 0.57 & 0.41 - 0.80 & 0.52 - 0.83 \end{pmatrix}. \quad (1)$$

Very little is known theoretically on how leptons perform their mixing of generations. To construct a theory of flavor in a ground-up approach one needs to first do a theoretical parametrization of the extracted numbers and then search a theory which can generate this pattern of mixing. Therefore we have to make a few ansatze first in order to proceed toward a fuller theory. A number of them already exists in the literature[20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 34, 35]. We have cited a few examples in Table 1 to motivate the present work.

At this point let us briefly discuss mixing in the lepton sector and fix the definition of the U matrix in the context of a $G \equiv SU(2)_L \times U(1)_Y$ model. Rotation in flavor space becomes visible in interactions between charge changing weak currents and a gauge boson (lets call mass eigenstates of gauge bosons A_μ and gauge eigenstates A_μ^0) after G symmetry is broken. In the G invariant form interactions of left handed gauge eigentates is diagonal in flavor space, and looks as following;

$$(\bar{l}_L^\alpha \gamma^\mu \nu_L^\alpha) A_\mu^0, \quad (2)$$

where $A_\mu^0, l_L^\alpha, \nu_L^\alpha$ are gauge eigenstates. When $SU(2)_L \times U(1)_Y$ symmetry is broken by the Higgs mechanism or extra-dimensional mechanism or some other yet-unknown mechanism, naturally gauge eigenstates no-longer exist, and the gauge bosons now become massive. Interaction given in Eqn 2, can now be easily rearranged in terms of mass eigenstates as,

$$(\bar{l}^i \gamma^\mu U_{ij} \nu^j) A_\mu. \quad (3)$$

We have defined unitary mixing matrix U of lepton sector in this conventional way given in Eqn. 3. We do not commit to any specific method of $SU(2)_L \times U(1)$ symmetry breaking and simply concentrate on a parametrization of the mixing matrix of lepton sector which will produce Eqn. 1 in a natural way.

If we invoke a specific mechanism of symmetry breaking or a specific flavor symmetry, present paper would be less generally valid and hence weaker.

Now we are ready to put forward the first ansatz, in which the input parameter ϕ may either be real or complex. When the input parameter is real from one input we will get three mixing angles, and when ϕ is complex, from two real inputs (a complex number can be thought of two real numbers), we will predict three mixing angles and the CP violating parameter J . In both cases, there are two predictions. However we will presently see that complex values of ϕ will lead to violation of unitarity of the matrix U_0 so the predictions are not dependable for complex ϕ . Therefore we will confine ourselves to real ϕ .

HYPERBOLIC ANSATZ 1

$$U^0 = \begin{pmatrix} -\text{sech } \phi & \tanh \phi & 0 \\ \frac{\tanh \phi}{\sqrt{2}} & \frac{\text{sech } \phi}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{\tanh \phi}{\sqrt{2}} & \frac{\text{sech } \phi}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \quad |\phi| \approx \pi/6. \quad (4)$$

The results for the hyperbolic ansatz 1 are given in Table 2. We see that depending on angle ϕ , the quantity $\sin^2 2\theta_{solar}$ changes, whereas $\sin^2 2\theta_{atm}$ remain strictly maximal at 1.0 for all ϕ . U_{e3} is also strictly zero for all ϕ . Therefore this ansatz is in very good agreement with currently favored experimental values. Note that ϕ is not necessarily real. When ϕ takes real values so, $J = 0$, the results are given in Table 2.

One may wonder whether U^0 is unitary? To answer that question let us calculate,

$$U^{0\dagger} U^0 = \begin{pmatrix} A & B & 0 \\ B & A & 0 \\ 0 & 0 & A \end{pmatrix} \quad (5)$$

Where $A = \text{sech}^2 \phi + \tanh^2 \phi$, $B = \text{sech } \phi (\tanh \phi)^* - \tanh \phi (\text{sech } \phi)^*$. One can check analytically or using MATHEMATICA [53] that $B = 0$ and $A = 1$ condition is satisfied for real ϕ . For general complex ϕ we get that $A \neq 1$ and $B \neq 0$. Therefore we confine ourselves only to real ϕ .

HYPERBOLIC ANSATZ 2

$$U = \begin{pmatrix} -\text{sech } \phi \cos \delta & \tanh \phi & \text{sech } \phi \sin \delta \\ \frac{\sin \delta + \cos \delta}{\sqrt{2}} \tanh \phi & \frac{\text{sech } \phi}{\sqrt{2}} & \frac{\cos \delta - \sin \delta}{\sqrt{2}} \tanh \phi \\ \frac{\sin \delta + \cos \delta}{\sqrt{2}} \tanh \phi & \frac{\text{sech } \phi}{\sqrt{2}} & -\frac{\cos \delta + \sin \delta}{\sqrt{2}} \tanh \phi \end{pmatrix} \quad \lim_{\delta \rightarrow 0} U = U^0. \quad (6)$$

The results for the hyperbolic ansatz 2 are given in Table 3. We see that U_{e3} is small for small δ . Due to unitarity relation of the e'th row $U_{e1}^2 + U_{e2}^2 + U_{e3}^2 = 1$, this accentuation will alter the solar and atmospheric mixing angles slightly. Therefore the atmospheric mixing is not strictly maximal any more in this case, however $\sin^2 2\theta_{atm}$ remains very close to 1, $\sin^2 2\theta_{solar}$ remains well within the Large Mixing Angle (LMA) solution region.

One can again question whether the matrix U is unitary. The answer is that U can be factorised as,

$$U = U^0 \times O_{13}, \quad (7)$$

where,

$$O_{13} = \begin{pmatrix} \cos \delta & 0 & \sin \delta \\ 0 & 1 & 0 \\ -\sin \delta & 0 & \cos \delta \end{pmatrix}. \quad (8)$$

Because U^0 and O_{13} are unitary matrices, we find that U is unitary.

Now let us discuss whether these ansatze will lead to CP violation. For ansatz 1 and 2, let the Jarlskogian invariant[36, 37, 38, 39, 40] given by $\text{Im} [U_{e1}U_{\mu 2}U_{e2}^*U_{\mu 1}^*]$, which vanishes if and only if CP is conserved, be denoted by J_1 and J_2 . Then we get,

$$J_1 = -\frac{1}{2} \text{Im} [(\text{sech } \phi)^2 ((\tanh \phi)^*)^2] \quad (9)$$

$$J_2 = -\frac{1}{2} \text{Im}[(\text{sech } \phi)^2 \cos \delta (\sin \delta + \cos \delta \tanh \phi)^* (\tanh \phi)^*]. \quad (10)$$

So we see that ansatz 2 may violate CP in principle and ansatz 1 may also violate it for complex ϕ . Because we are considering only real ϕ ansatz 1 will give $J_1 = 0$ conserving CP. Let us take a sample complex value of $\delta = 0.01 + 0.01 i$ along with a real $\phi = 0.55$, then we get four predictions, $\sin^2 2\theta_{solar} = 0.75$, $\sin^2 2\theta_{atm} = 0.99$, $\sin^2 2\theta_{e3} = 0.00059$ and $J = 0.0000132$. Therefore a small but non-zero values of $\sin^2 2\theta_{e3}$ as well as J are obtained in ansatz 2, allowing it to violate CP. Predictions of Ansatz 2 will smoothly carry over to those of Ansatz 1 in the limit $\delta \rightarrow 0$ where there is necessarily no CP violation. Furthermore small values of δ are needed to get correct solar and atmospheric mixing in ansatz 2.

Following relationships of hyperbolic functions with circular functions may be useful in linking our ansatze with more conventional ansatze using

circular functions.

$$\operatorname{sech} \phi = \sec(i\phi) \quad (11)$$

$$\tanh \phi = -i \tan(i\phi) \quad (12)$$

For purely imaginary ϕ one can immediately make a correspondence with more conventional pictures using circular functions. Because ϕ need not be purely imaginary it is not immediately apparent how to interpret this insatze in terms of simple rotation angles and corresponding circular functions such as Sine and Cosine.

We must clarify what have we meant by the term “Prediction”. In any model of physical systems, if there are N measurable outputs for $(N-n)$ input parameters, the model is said to have ‘ n ’ “predictions”. Predictions are important from the point of view of falsifiability. Suppose in experimental measurements we find $U_{e3} \neq 0$, then our HYPERBOLIC ANSATZ 1 will be proven to be false. If we have N inputs for N outputs then one can always adjust N inputs to correctly reproduce N experimental outputs. In this case the physical model can never be proven wrong. Therefore such physical models will not be interesting.

To conclude, in this article we have put forward two ansatze for lepton mixing. These ansatze use hyperbolic functions for which arguements may either be real or complex. To preserve unitarity of mixing matrix we should keep ϕ real. To the best of our knowledge parametrizations of lepton mixing using hyperbolic functions do not exist in literature. Present parametrization is capable of predicting correct values for solar, atmospheric and CHOOZ[41, 42, 43, 44, 45] angles for natural values of parameters. Such parametrizations for the lepton mixing matrix are important in the sense that they constitute a first step toward building more complete theories of flavor physics. From experimental point of view a good number of new experiments such as DOUBLE-CHOOZ[46], new reactor experiments[47, 48], as well as long-baseline experiments[49, 50, 51, 52] are either coming up or being proposed, those will be used to measure angles including θ_{e3} , where our ansatze can undergo rigorous testing and verification.

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<u>The ω mixing</u>	$U = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \\ 1 & 1 & 1 \end{pmatrix}$
<u>Bi-maximal</u>	$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix}$
<u>Zee</u>	$U = \begin{pmatrix} -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$
<u>Giunti</u>	$U = \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ \frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$
<u>sine-cosine</u>	$U = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\frac{\sin \theta}{\sqrt{2}} & \frac{\cos \theta}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{\sin \theta}{\sqrt{2}} & -\frac{\cos \theta}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$
<u>Tri-maximal</u>	$U = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$
<u>Minakata-Yasuda/Fritzsch and Zing</u>	$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & -\frac{i}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$

Table 1: A selection of a few ansatze of lepton mixing that exist in literature. This selection is only a sample, not a complete list.

ϕ	$\sin^2 2\theta_{solar}$	$\sin^2 2\theta_{atm}$	$\sin^2 2\theta_{e3}$	J
0.50	0.6717	1.0	0	0
0.52	0.7044	1.0	0	0
0.54	0.7358	1.0	0	0
0.56	0.7658	1.0	0	0
0.58	0.7942	1.0	0	0
0.60	0.8209	1.0	0	0

Table 2: Results for the hyperbolic ansatz 1

ϕ	δ	$\sin^2 2\theta_{solar}$	$\sin^2 2\theta_{atm}$	$\sin^2 2\theta_{e3}$	J
0.50	0.1	0.6756	0.9914	0.0311	0
0.55	0.1	0.7547	0.9899	0.0296	0
0.60	0.1	0.8244	0.9884	0.0281	0
0.50	0.01	0.6718	0.9999	0.00031	0
0.55	0.01	0.7510	0.9999	0.00029	0
0.60	0.01	0.8209	0.9998	0.00028	0

Table 3: Results for the hyperbolic ansatz 2.